ω_1 under Π_1-Collection

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Abstract

We describe a proof-theoretic bound on Σ_2-definable countable ordinals in Kripke-Platek set theory with Π_1-Collection and the existence of ω_1.

1 Introduction

Let (ω_1) denote an axiom stating that 'there exists an uncountable regular ordinal', and $T_1 := \mathsf{KP}\omega + (V = L) + (\Pi_1\text{-Collection}) + (\omega_1)$. Let ρ_0 denote the least ordinal above ω_1 such that $L_{\rho_0} \models (\Pi_1\text{-Collection})$. In this note a collapsing function $\Psi_{\omega_1} : \alpha \mapsto \Psi_{\omega_1}(\alpha) < \omega_1$ is defined, and it is shown that for each $n < \omega$, $T_1 \vdash \forall\alpha < \omega_n(\rho_0 + 1)\exists x < \omega_1(x = \Psi_{\omega_1}(\alpha))$ with a Σ_2-formula $x = \Psi_{\omega_1}(\alpha)$, cf. Lemma 4.5. Conversely we show the

Theorem 1.1 *For a sentence $\exists x \in L_{\omega_1}\, \varphi(x)$ with a Σ_2-formula $\varphi(x)$, if*

$$T_1 \vdash \exists x \in L_{\omega_1}\, \varphi(x)$$

then

$$\exists n < \omega[T_1 \vdash \exists x \in L_{\Psi_{\omega_1}(\omega_n(\rho_0+1))}\varphi(x)].$$

This paper relies on our [1].

2 Σ_1-Skolem hulls

Everything in this section is reproduced from [1].

For a model $\langle M; \in\restriction(M \times M)\rangle$ and $X \subset M$, $\Sigma_1^M(X)$ denotes the set of $\Sigma_1(X)$-definable subsets of M, where $\Sigma_1(X)$-formulae may have parameters from X. $\Sigma_1^M(M)$ is denoted $\Sigma_1(M)$.

An ordinal $\alpha > 1$ is said to be a *multiplicative principal number* iff α is closed under ordinal multiplication, i.e., $\exists\beta[\alpha = \omega^{\omega^\beta}]$. If α is a multiplicative principal number, then α is closed under Gödel's pairing function j and there exists a Δ_1-bijection between α and L_α for the constructible hierarchy L_α up to α. In this section σ is assumed to be a multiplicative principal number$> \omega$.

Definition 2.1 *1.* $cf(\kappa) := \min\{\alpha \leq \kappa : \text{there is a cofinal map } f : \alpha \to \kappa\}$.

2. $\rho(L_\sigma)$ denotes the Σ_1-*projectum* of L_σ: $\rho(L_\sigma)$ is the least ordinal ρ such that $\mathcal{P}(\rho) \cap \Sigma_1(L_\sigma) \not\subset L_\sigma$.

3. Let $\alpha \leq \beta$ and $f : L_\alpha \to L_\beta$. Then the map f is a Σ_1-*elementary embedding*, denoted $f : L_\alpha \prec_{\Sigma_1} L_\beta$ iff for any $\Sigma_1(L_\alpha)$-sentence $\varphi[\bar{a}]$ ($\bar{a} \subset L_\alpha$), $L_\alpha \models \varphi[\bar{a}] \Leftrightarrow L_\beta \models \varphi[f(\bar{a})]$ where $f(\bar{a}) = f(a_1), \dots, f(a_k)$ for $\bar{a} = a_1, \dots, a_k$. An ordinal γ such that $\forall \delta < \gamma [f(\delta) = \delta] \,\&\, f(\gamma) > \gamma$ is said to be the *critical point* of the Σ_1- elementary embedding f if such an ordinal γ exists.

4. For $X \subset L_\sigma$, $\mathrm{Hull}^\sigma_{\Sigma_1}(X)$ denotes the set (Σ_1-*Skolem hull* of X in L_σ) defined as follows. $<_L$ denotes a Δ_1-well ordering of the constructible universe L. Let $\{\varphi_i : i \in \omega\}$ denote an enumeration of Σ_1-formulae in the language $\{\in\}$. Each is of the form $\varphi_i \equiv \exists y \theta_i(x, y; u)$ ($\theta \in \Delta_0$) with fixed variables x, y, u. Set for $b \in X$

$$\begin{array}{rcl} r^\sigma_{\Sigma_1}(i,b) & \simeq & \text{the } <_L\text{-least } c \in L_\sigma \text{ such that } L_\sigma \models \theta_i((c)_0, (c)_1; b) \\ h^\sigma_{\Sigma_1}(i,b) & \simeq & (r^\sigma_{\Sigma_1}(i,b))_0 \\ \mathrm{Hull}^\sigma_{\Sigma_1}(X) & = & rng(h^\sigma_{\Sigma_1}) = \{h^\sigma_{\Sigma_1}(i,b) \in L_\sigma : i \in \omega, b \in X\} \end{array}$$

Then $L_\sigma \models \exists x \exists y\, \theta_i(x, y; b) \to h^\sigma_{\Sigma_1}(i,b) \downarrow \,\&\, \exists y\, \theta_i(h^\sigma_{\Sigma_1}(i,b), y; b)$.

Proposition 2.2 *Assume that X is a set in L_σ. Then $r^\sigma_{\Sigma_1}$ and $h^\sigma_{\Sigma_1}$ are partial $\Delta_1(L_\sigma)$-maps such that the domain of $h^\sigma_{\Sigma_1}$ is a $\Sigma_1(L_\sigma)$-subset of $\omega \times X$. Therefore its range* $\mathrm{Hull}^\sigma_{\Sigma_1}(X)$ *is a $\Sigma_1(L_\sigma)$-subset of L_σ.*

Proposition 2.3 *Let $Y = \mathrm{Hull}^\sigma_{\Sigma_1}(X)$. For any $\Sigma_1(Y)$-sentence $\varphi(\bar{a})$ with parameters $\bar{a}$ from Y $L_\sigma \models \varphi(\bar{a}) \Leftrightarrow Y \models \varphi(\bar{a})$. Namely $Y \prec_{\Sigma_1} L_\sigma$.*

Definition 2.4 (Mostowski collapsing function F)

By Proposition 2.3 and the Condensation Lemma we have an isomorphism (Mostowski collapsing function)

$$F : \mathrm{Hull}^\sigma_{\Sigma_1}(X) \leftrightarrow L_\gamma$$

for an ordinal $\gamma \leq \sigma$ such that $F \restriction Y = id \restriction Y$ for any transitive $Y \subset \mathrm{Hull}^\sigma_{\Sigma_1}(X)$.

Let us denote, though $\sigma \notin dom(F) = \mathrm{Hull}^\sigma_{\Sigma_1}(X)$

$$F(\sigma) := \gamma.$$

Also for the above Mostowski collapsing map F let

$$F^{\Sigma_1}(x; \sigma, X) := F(x).$$

The inverse $G := F^{-1}$ of F is a Σ_1-elementary embedding from $L_{F(\sigma)}$ to L_σ.

Proposition 2.5 *Let $L_\sigma \models \mathsf{KP}\omega + \Sigma_1$-Collection. Then for $\kappa \leq \sigma$, $\{(x, y) : x < \kappa \,\&\, y = \min\{y < \kappa : \mathrm{Hull}^{\sigma}_{\Sigma_1}(x \cup \{\kappa\}) \cap \kappa \subset y\}\}$ is a $Bool(\Sigma_1(L_\sigma))$-predicate on κ, and hence a set in L_σ if $\kappa < \sigma$ and $L_\sigma \models \Sigma_1$-Separation.*

$F^{\Sigma_1}_{x\cup\{\kappa\}}(y)$ denotes the Mostowski collapse $F^{\Sigma_1}(y; \sigma, x \cup \{\kappa\})$.

Theorem 2.6 *Let σ be an ordinal such that $L_\sigma \models \mathrm{KP}\omega + \Sigma_1$-Separation, and $\omega \leq \alpha < \kappa < \sigma$ with α a multiplicative principal number and κ a limit ordinal. Then the following conditions are mutually equivalent:*

1. *$L_\sigma \models {}^{\alpha}\kappa \subset L_\kappa$.*
2. *$L_\sigma \models \alpha < cf(\kappa)$.*
3. *There exists an ordinal x such that $\alpha < x < \kappa$, $\mathrm{Hull}^{\sigma}_{\Sigma_1}(x \cup \{\kappa\}) \cap \kappa \subset x$ and $F^{\Sigma_1}_{x\cup\{\kappa\}}(\sigma) < \kappa$.*
4. *For the Mostowski collapse $F^{\Sigma_1}_{x\cup\{\kappa\}}(y)$, there exists an ordinal x such that $\alpha < x = F^{\Sigma_1}_{x\cup\{\kappa\}}(\kappa) < F^{\Sigma_1}_{x\cup\{\kappa\}}(\sigma) < \kappa$, and for any Σ_1-formula φ and any $a \in L_x$, $L_\sigma \models \varphi[\kappa, a] \to L_{F^{\Sigma_1}_{x\cup\{\kappa\}}(\sigma)} \models \varphi[x, a]$ holds.*

Definition 2.7 $T_1 := \mathsf{KP}\omega + (V = L) + (\Pi_1\text{-Collection}) + (\omega_1)$ denotes an extension of the Kripke-Platek set theory with the axioms of infinity, constructibility, Π_1-Collection and the following axiom:

$$(\omega_1) \exists\kappa\forall\alpha < \kappa\exists\beta, \gamma < \kappa[\alpha < \beta < \gamma \land L_\gamma = rng(F^{\Sigma_1}_{\beta\cup\{\kappa\}}) \land \mathrm{Hull}_{\Sigma_1}(\beta \cup \{\kappa\}) \cap \kappa \subset \beta]$$

where $F^{\Sigma_1}_{\beta\cup\{\kappa\}} : \mathrm{Hull}_{\Sigma_1}(\beta \cup \{\kappa\}) \to L_\gamma$ is the Mostowski collapsing map, and $\mathrm{Hull}_{\Sigma_1}(x)$ is the Σ_1-Skolem hull of sets x in the universe.

From Theorem 2.6 we see that $T_1 \vdash \exists\kappa\forall\alpha < \kappa(\alpha < cf(\kappa))$.

3 A theory equivalent to T_1

Referring Theorem 2.6 let us interpret T_1 to another theory. The base language here is $\{\in\}$.

Let ρ_0 denotes the least ordinal above the least uncountable ordinal ω_1 such that $L_{\rho_0} \models (\Pi_1\text{-Collection})$. $F_X(x) := F^{\Sigma_1}(x; \rho_0, X)$ and $\mathrm{Hull}(X) := \mathrm{Hull}^{\rho_0}_{\Sigma_1}(X)$.

The predicate P is intended to denote the relation $P(x, y)$ iff $x = F_{x\cup\{\omega_1\}}(\omega_1)$ and $y = F_{x\cup\{\omega_1\}}(\rho_0)$. Also the predicate $P_{\rho_0}(x)$ is intended to denote the relation $P_{\rho_0}(x)$ iff $x = F_x(\rho_0)$.

Definition 3.1 $\mathrm{T}(\omega_1)$ denotes the set theory defined as follows.

1. Its language is $\{\in, P, P_{\rho_0}, \omega_1\}$ for a binary predicate P, a unary predicate P_{ρ_0} and an individual constant ω_1.

2. Its axioms are obtained from those of $\mathsf{KP}\omega + (\Pi_1\text{-Collection})$ in the expanded language [1], the axiom of constructibility $V = L$ together with the axiom schema saying that ω_1 is an uncountable regular ordinal, cf. (2) and (1), and if $P(x,y)$ then x is a critical point of the Σ_1-elementary embedding from $L_y \cong \mathrm{Hull}(x \cup \{\omega_1\})$ to the universe L_{ρ_0}, cf. (1), and if $P_{\rho_0}(x)$ then x is a critical point of the Σ_1-elementary embedding from $L_x \cong \mathrm{Hull}(x)$ to the universe L_{ρ_0}, cf.(3): for a formula φ and an ordinal α, φ^α denotes the result of restricting every unbounded quantifier $\exists z, \forall z$ in φ to $\exists z \in L_\alpha, \forall z \in L_\alpha$.

   (a) $x \in Ord$ is a Δ_0-formula saying that 'x is an ordinal'.
   $(\omega < \omega_1 \in Ord)$, $(P(x,y) \to \{x,y\} \subset Ord \land x < y < \omega_1)$ and $(P_{\rho_0}(x) \to x \in Ord)$.

   (b)
   $$P(x,y) \to a \in L_x \to \varphi[\omega_1, a] \to \varphi^y[x,a] \tag{1}$$
   for any Σ_1-formula φ in the language $\{\in\}$.

   .

   (c)
   $$a \in Ord \cap \omega_1 \to \exists x, y \in Ord \cap \omega_1 [a < x \land P(x,y)] \tag{2}$$

   (d)
   $$P_{\rho_0}(x) \to a \in L_x \to \varphi[a] \to \varphi^x[a] \tag{3}$$
   for any Σ_1-formula φ in the language $\{\in\}$.

   (e)
   $$a \in Ord \to \exists x \in Ord[a < x \land P_{\rho_0}(x)] \tag{4}$$

Remark. Though the axioms (3) and (4) for the Π_1-definable predicate $P_{\rho_0}(x)$ are derivable from Π_1-Collection, the primitive predicate symbol $P_{\rho_0}(x)$ is useful for our prof-theoretic study, cf. the proof of Lemma 5.20 below.

Lemma 3.2 $\mathrm{T}(\omega_1)$ *is a conservative extension of the set theory T_1.*

Proof. First consider the axioms of T_1 in $T(\omega_1)$. The axiom (ω_1) follows from (1). Hence we have shown that T_1 is contained in $\mathrm{T}(\omega_1)$.

Next we show that $\mathrm{T}(\omega_1)$ is interpretable in T_1. Let κ be an ordinal in the axiom (ω_1). Interpret the predicate $P(x,y) \leftrightarrow \{x,y\} \subset Ord \land (\mathrm{Hull}(x \cup \{\kappa\}) \cap \kappa \subset x) \land (y = \sup\{F_{x\cup\{\kappa\}}(a) : a \in \mathrm{Hull}(x \cup \{\kappa\})\})$. We see from Theorem 2.6 that the interpreted (1) and (2) are provable in T_1.

It remains to show the interpreted (3) and (4) in T_1. It suffices to show that given an ordinal α, there exists an ordinal $x > \alpha$ such that $\mathrm{Hull}(x) \cap Ord \subset x$.

First we show that for any α there exists a β such that $\mathrm{Hull}(\alpha) \cap Ord \subset \beta$. By Proposition 2.2 let $h^{\rho_0}_{\Sigma_1}$ be the Δ_1-surjection from the Σ_1-subset $dom(h^{\rho_0}_{\Sigma_1})$ of

[1] This means that the predicates P, P_{ρ_0} do not occur in Δ_0-formulae for Δ_0-Separation and Π_1-formulae Π_1-Collection.

$\omega\times\alpha$ to Hull(α), which is a Σ_1-class. From Σ_1-Separation we see that $dom(h^{\rho_0}_{\Sigma_1})$ is a set. Hence by Σ_1-Collection, Hull$(\alpha) = rng(h^{\rho_0}_{\Sigma_1})$ is a set. Therefore the ordinal sup(Hull$(\alpha)\cap Ord$) exists in the universe.

As in Proposition 2.5 we see that $X = \{(\alpha,\beta) : \beta = \min\{\beta\in Ord : \text{Hull}(\alpha)\cap Ord \subset \beta\}\}$ is a set in L_{ρ_0} as follows. Let $\varphi(\beta)$ be the Π_1-predicate $\varphi(\beta) :\Leftrightarrow \forall\gamma\in Ord[\gamma\in \text{Hull}(\alpha)\to\gamma\in\beta]$. Then $\beta = \min\{\beta : \text{Hull}(\alpha)\cap Ord\subset\beta\}$ iff $\varphi(\beta)\land\forall\gamma<\beta\neg\varphi(\gamma)$, which is $Bool(\Sigma_1(L_{\rho_0}))$ by Π_0-Collection. Hence X is a set in L_{ρ_0}.

Define recursively ordinals $\{x_n\}_n$ as follows. $x_0 = \alpha+1$, and x_{n+1} is defined to be the least ordinal x_{n+1} such that Hull$(x_n)\cap Ord\subset x_{n+1}$, i.e., $(x_n, x_{n+1})\in X$. We see inductively that such an ordinal x_n exists. Moreover $n\mapsto x_n$ is a Δ_1-map. Then $x = \sup_n x_n < \rho_0$ is a desired one. □

4 Ordinals for ω_1

For our proof-theoretic analysis of T_1, we need to talk about 'ordinals' less than the next epsilon number to the order type of the class of ordinals inside T_1. Let $Ord^\varepsilon\subset V$ and $<^\varepsilon$ be Δ-predicates such that for any transitive and wellfounded model V of $\mathsf{KP}\omega$, $<^\varepsilon$ is a well ordering of type ε_{ρ_0+1} on Ord^ε for the order type ρ_0 of the class Ord in V. $<^\varepsilon$ is seen to be a canonical ordering as stated in the following Proposition 4.1.

Proposition 4.1 *1. $\mathsf{KP}\omega$ proves the fact that $<^\varepsilon$ is a linear ordering.*

2. For any formula φ and each $n<\omega$,

$$\mathsf{KP}\omega\vdash\forall x\in Ord^\varepsilon(\forall y<^\varepsilon x\,\varphi(y)\to\varphi(x))\to\forall x<^\varepsilon\omega_n(\rho_0+1)\varphi(x) \quad (5)$$

In what follows of this section we work in T_1. For simplicity let us identify the code $x\in Ord^\varepsilon$ with the 'ordinal' coded by x, and $<^\varepsilon$ is denoted by $<$ when no confusion likely occurs. Note that the ordinal ρ_0 is the order type of the class of ordinals in the intended model L_{ρ_0} of T_1. Define simultaneously the classes $\mathcal{H}_\alpha(X)\subset L_{\rho_0}\cup\varepsilon_{\rho_0+1}$ and the ordinals $\Psi_{\omega_1}(\alpha)$ and $\Psi_{\rho_0}(\alpha)$ for $\alpha<^\varepsilon\varepsilon_{\rho_0+1}$ and sets $X\subset L_{\omega_1}$ as follows. We see that $\mathcal{H}_\alpha(X)$ and $\Psi_\kappa(\alpha)\,(\kappa\in\{\omega_1,\rho_0\})$ are (first-order) definable as a fixed point in T_1, cf. Proposition 4.4.

Recall that $\text{Hull}(X) = \text{Hull}^{\rho_0}_{\Sigma_1}(X)\subset L_{\rho_0}$ and $F_X(x) = F^{\Sigma_1}(x;\rho_0,X)$ with $F_X : \text{Hull}(X)\to L_\gamma$ for $X\subset L_{\rho_0}$ and a $F_X(\rho_0)=\gamma\le\rho_0$.

Definition 4.2 $\mathcal{H}_\alpha(X)$ is the Skolem hull of $\{0,\omega_1,\rho_0\}\cup X$ under the functions $+,\alpha\mapsto\omega^\alpha,\Psi_{\omega_1}\restriction\alpha,\Psi_{\rho_0}\restriction\alpha$, the Σ_1-definability, and the Mostowski collapsing functions $(x,d)\mapsto F_{x\cup\{\omega_1\}}(d)$ (Hull$(x\cup\{\omega_1\})\cap\omega_1\subset x$) and $d\mapsto F_x(d)$ (Hull$(x)\cap\rho_0\subset x$).

1. $\{0,\omega_1,\rho_0\}\cup X\subset\mathcal{H}_\alpha(X)$.

2. $x,y\in\mathcal{H}_\alpha(X)\Rightarrow x+y,\omega^x\in\mathcal{H}_\alpha(X)$.

3. $\gamma\in\mathcal{H}_\alpha(X)\cap\alpha\Rightarrow\Psi_\kappa(\gamma)\in\mathcal{H}_\alpha(X)$ for $\kappa\in\{\omega_1,\rho_0\}$.

4. $\mathrm{Hull}(\mathcal{H}_\alpha(X) \cap L_{\rho_0}) \subset \mathcal{H}_\alpha(X)$.

   Namely for any Σ_1-formula $\varphi[x, \vec{y}]$ in the language $\{\in\}$ and parameters $\vec{a} \subset \mathcal{H}_\alpha(X) \cap L_{\rho_0}$, if $b \in L_{\rho_0}$, $(L_{\rho_0}, \in) \models \varphi[b, \vec{a}]$ and $(L_{\rho_0}, \in) \models \exists! x\, \varphi[x, \vec{a}]$, then $b \in \mathcal{H}_\alpha(X)$.

5. If $x \in \mathcal{H}_\alpha(X) \cap \omega_1$ with $\mathrm{Hull}(x \cup \{\omega_1\}) \cap \omega_1 \subset x$, and $d \in (\mathrm{Hull}(x \cup \{\omega_1\}) \cap \mathcal{H}_\alpha(X)) \cup \{\rho_0\}$, then $F_{x \cup \{\omega_1\}}(d) \in \mathcal{H}_\alpha(X)$.

6. If $x \in \mathcal{H}_\alpha(X) \cap \rho_0$ with $\mathrm{Hull}(x) \cap \rho_0 \subset x$, and $d \in (\mathrm{Hull}(x) \cap \mathcal{H}_\alpha(X)) \cup \{\rho_0\}$, then $F_x(d) \in \mathcal{H}_\alpha(X)$.

For $\kappa \in \{\omega_1, \rho_0\}$

$$\Psi_\kappa(\alpha) := \min\{\beta \leq \kappa : \mathcal{H}_\alpha(\beta) \cap \kappa \subset \beta\}.$$

The ordinal $\Psi_\kappa(\alpha)$ is well defined and $\Psi_\kappa(\alpha) \leq \kappa$ for $\kappa \in \{\omega_1, \rho_0\}$.

Proposition 4.3 *1. $\mathcal{H}_\alpha(X)$ is closed under Σ_1-definability: $\vec{a} \subset \mathcal{H}_\alpha(X) \cap L_{\rho_0} \Rightarrow \mathrm{Hull}(\vec{a}) \subset \mathcal{H}_\alpha(X)$.*

2. $\mathrm{Hull}(\Psi_{\omega_1}(\alpha) \cup \{\omega_1\}) \cap \omega_1 = \Psi_{\omega_1}(\alpha)$ and $\mathrm{Hull}(\Psi_{\rho_0}(\alpha)) \cap \rho_0 = \Psi_{\rho_0}(\alpha) > \omega_1$

3. $\mathcal{H}_\alpha(X)$ is closed under the Veblen function φ on ρ_0, $x, y \in \mathcal{H}_\alpha(X) \cap \rho_0 \Rightarrow \varphi xy \in \mathcal{H}_\alpha(X)$.

4. If $x \in \mathcal{H}_\alpha(X) \cap \omega_1$, $\mathrm{Hull}(x \cup \{\omega_1\}) \cap \omega_1 \subset x$, and $\delta \in (\mathrm{Hull}(x \cup \{\omega_1\}) \cap \mathcal{H}_\alpha(X)) \cup \{\rho_0\}$, then $F_{x \cup \{\omega_1\}}(\delta) \in \mathcal{H}_\alpha(X)$.

5. If $x \in \mathcal{H}_\alpha(X) \cap \rho_0$, $\mathrm{Hull}(x) \cap \rho_0 \subset x$, and $\delta \in (\mathrm{Hull}(x) \cap \mathcal{H}_\alpha(X)) \cup \{\rho_0\}$, then $F_x(\delta) \in \mathcal{H}_\alpha(X)$.

The following Proposition 4.4 is easy to see.

Proposition 4.4 *Both of $x = \mathcal{H}_\alpha(X)$ and $y = \Psi_\kappa(\alpha)\,(\kappa \in \{\omega_1, \rho_0\})$ are Σ_2-predicates as fixed points in $\mathsf{KP}\omega$.*

Lemma 4.5 *For each $n < \omega$,*

$$T_1 \vdash \forall \alpha < \omega_{n+1}(\rho_0 + 1) \forall \kappa \in \{\omega_1, \rho_0\} \exists x < \kappa [x = \Psi_\kappa(\alpha)].$$

Proof. Let $\kappa \in \{\omega_1, \rho_0\}$. By Proposition 4.4 both $x = \mathcal{H}_\alpha(\beta)$ and $y = \Psi_\kappa(\alpha)$ are Σ_2-predicates. We show that $A(\alpha) :\Leftrightarrow \forall \beta < \rho_0 \exists x [x = \mathcal{H}_\alpha(\beta)] \wedge \forall \kappa \in \{\omega_1, \rho_0\} \exists \beta < \kappa [\Psi_\kappa(\alpha) = \beta]$ is progressive. Then $\forall \alpha < \omega_{n+1}(\rho_0 + 1) \forall \kappa \in \{\omega_1, \rho_0\} \exists x < \kappa [x = \Psi_\kappa(\alpha)]$ will follow from transfinite induction up to $\omega_{n+1}(\rho_0 + 1)$, cf. (5) in Proposition 4.1.

Assume $\forall \gamma < \alpha\, A(\gamma)$ as our IH. Since $dom(h^{\rho_0}_{\Sigma_1})$ is a Σ_1-subset of $\omega \times \beta$ for $\beta < \rho_0$, it is a set by Σ_1-Separation. Then so is the image $\mathrm{Hull}(\beta)$ of the Δ_1-map $h^{\rho_0}_{\Sigma_1}$. Hence $\forall \beta < \rho_0 \exists h [h = \mathrm{Hull}(\beta)]$.

We see from this, IH and Σ_2-Collection that $\forall\beta < \rho_0 \exists x[x = \mathcal{H}_\alpha(\beta) = \bigcup_m \mathcal{H}^m_\alpha(\beta)]$, where $\mathcal{H}^m_\alpha(\beta)$ is an m-th stage of the construction of $\mathcal{H}_\alpha(\beta)$ such that $x = \mathcal{H}^m_\alpha(\beta)$ is a Σ_2-predicate.

Define recursively ordinals $\{\beta_m\}_m$ for $\kappa \in \{\omega_1, \rho_0\}$ as follows. $\beta_0 = 0$, and β_{m+1} is defined to be the least ordinal $\beta_{m+1} \leq \kappa$ such that $\mathcal{H}_\alpha(\beta_m) \cap \kappa \subset \beta_{m+1}$.

We see inductively that $\beta_m < \kappa$ using the regularity of κ and the facts that $\forall\beta < \kappa \exists x[x = \mathcal{H}_\alpha(\beta) \land card(x) < \kappa]$, where $card(x) < \kappa$ designates that there exists a surjection $f : \gamma \to x$ for a $\gamma < \kappa$ and $f \in L_{\rho_0}$. Moreover $m \mapsto \beta_m$ is a Σ_2-map. Therefore $\beta = \sup_m \beta_m < \kappa$ enjoys $\mathcal{H}_\alpha(\beta) \cap \kappa \subset \beta$. □

5 Operator controlled derivations for T_1

5.1 An intuitionistic fixed point theory $\mathbf{FiX}^i(T_1)$

Let us introduce an intuitionistic fixed point theory $\mathrm{FiX}^i(T_1)$ over the set theory T_1. Fix an X-strictly positive formula $\mathcal{Q}(X, x)$ in the language $\{\in, =, X\}$ with an extra unary predicate symbol X. In $\mathcal{Q}(X, x)$ the predicate symbol X occurs only strictly positive. The language of $\mathrm{FiX}^i(T_1)$ is $\{\in, =, Q\}$ with a fresh unary predicate symbol Q. The axioms in $\mathrm{FiX}^i(T_1)$ consist of the following:

1. All derivable sentences in T_1 in the language $\{\in\}$.
2. Induction schema for any formula φ in $\{\in, =, Q\}$:
   $\forall x(\forall y \in x\, \varphi(y) \to \varphi(x)) \to \forall x\, \varphi(x)$.
3. Fixed point axiom: $\forall x[Q(x) \leftrightarrow \mathcal{Q}(Q, x)]$.

The underlying logic in $\mathrm{FiX}^i(T_1)$ is defined to be the intuitionistic first-order predicate logic with equality.

Lemma 5.1 *Let $<^\varepsilon$ denote a Δ_1-predicate defined in section 4. For each $n < \omega$ and each formula φ in $\{\in, =, Q\}$,*

$$\mathrm{FiX}^i(T_1) \vdash \forall x(\forall y <^\varepsilon x\, \varphi(y) \to \varphi(x)) \to \forall x <^\varepsilon \omega_n(\rho_0 + 1)\varphi(x).$$

The following Theorem 5.2 is shown in [2].

Theorem 5.2 $\mathrm{FiX}^i(T_1)$ *is a conservative extension of* T_1.

5.2 Classes of formulae

In this subsection we work in T_1.

The language $\mathcal{L}_c$ is obtained from $\{\in, P, P_{\rho_0}, \omega_1\}$ by adding names (individual constants) c_a of each set $a \in L_{\rho_0}$. c_a is identified with a. A *term* in $\mathcal{L}_c$ is either a variable or a constant in L_{ρ_0}. Formulae in this language are defined in the next definition. Formulae are assumed to be in negation normal form.

Definition 5.3 1. Let $t_1, \ldots, t_m$ be terms. For each m-ary predicate constant $R \in \{\in, P, P_{\rho_0}\}$ $R(t_1, \ldots, t_m)$ and $\neg R(t_1, \ldots, t_m)$ are formulae, where $m = 1, 2$. These are called *literals.*

2. If A and B are formulae, then so are $A \land B$ and $A \lor B$.

3. Let t be a term. If A is a formula and the variable x does not occur in t, then $\exists x \in t\, A$ and $\forall x \in t\, A$ are formulae. $\exists x \in t\, A$, $\forall x \in t$ are *bounded quantifiers.*

4. If A is a formula and x a variable, then $\exists x\, A$ and $\forall x\, A$ are formulae. Unbounded quantifiers $\exists x, \forall x$ are denoted by $\exists x \in L_{\rho_0}, \forall x \in L_{\rho_0}$, resp.

For formulae A in $\mathcal{L}_c$, $\mathsf{qk}(A)$ denotes the finite set of sets $a \in L_{\rho_0}$ which are bounds of bounded quantifiers $\exists x \in a, \forall x \in a$ occurring in A. Moreover $\mathsf{k}(A)$ denotes the set of sets occurring in A. $\mathsf{k}(A)$ is defined to include bounds of bounded quantifiers. By definition we set $0 \in \mathsf{qk}(A)$. Thus $0 \in \mathsf{qk}(A) \subset \mathsf{k}(A) \subset L_{\rho_0}$.

Definition 5.4 1. $\mathsf{k}(\neg A) = \mathsf{k}(A)$ and similarly for qk.

2. $\mathsf{qk}(M) = \{0\}$ for any literal M.

3. $\mathsf{k}(Q(t_1, \ldots, t_m)) = (\{t_1, \ldots, t_m\} \cap L_{\rho_0}) \cup \{0\}$ for literals $Q(t_1, \ldots, t_m)$ with predicates Q in the set $\{\in, P, P_{\rho_0}\}$.

4. $\mathsf{k}(A_0 \lor A_1) = \mathsf{k}(A_0) \cup \mathsf{k}(A_1)$ and similarly for qk.

5. For unbounded quantifiers, $\mathsf{k}(\exists x\, A(x)) = \mathsf{k}(A(x))$ and similarly for qk.

6. For bounded quantifiers with $a \in L_{\rho_0}$, $\mathsf{k}(\exists x \in a\, A(x)) = \{a\} \cup \mathsf{k}(A(x))$ and similarly for qk.

7. For variables y, $\mathsf{k}(\exists x \in y\, A(x)) = \mathsf{k}(A(x))$ and similarly for qk.

8. For sets Γ of formulae $\mathsf{k}(\Gamma) := \bigcup\{\mathsf{k}(A) : A \in \Gamma\}$.

For example $\mathsf{qk}(\exists x \in a\, A(x)) = \{a\} \cup \mathsf{qk}(A(x))$ if $a \in L_{\rho_0}$.

Definition 5.5 For $a \in L_{\rho_0} \cup \{L_{\rho_0}\}$, $\mathrm{rk}_L(a)$ denotes the *L-rank* of a.

$$\mathrm{rk}_L(a) := \begin{cases} \min\{\alpha \in Ord : a \in L_{\alpha+1}\} & a \in L_{\rho_0} \\ \rho_0 & a = L_{\rho_0} \end{cases}$$

Definition 5.6 1. $A \in \Delta_0$ iff there exists a Δ_0-formula $\theta[\vec{x}]$ in the language $\{\in\}$ and terms $\vec{t}$ in $\mathcal{L}_c$ such that $A \equiv \theta[\vec{t}]$. This means that A is bounded, and the predicates P, P_{ρ_0} do not occur in A.

2. Putting $\Sigma_0 := \Pi_0 := \Delta_0$, the classes Σ_m and Π_m of formulae in the language $\mathcal{L}_c$ are defined as usual, where by definition $\Sigma_m \cup \Pi_m \subset \Sigma_{m+1} \cap \Pi_{m+1}$.

   Each formula in $\Sigma_m \cup \Pi_m$ is in prenex normal form with alternating unbounded quantifiers and Δ_0-matrix.

3. The set $\Sigma^{\Sigma_n}(\lambda)$ of sentences is defined recursively as follows. Let $\{a,b,c\}\subset L_{\rho_0}$ and $d\in L_{\rho_0}\cup\{L_{\rho_0}\}$.

   (a) Each Σ_n-sentence is in $\Sigma^{\Sigma_n}(\lambda)$.

   (b) Each literal including $Reg(a), P(a,b,c), P_{I,n}(a)$ and its negation is in $\Sigma^{\Sigma_n}(\lambda)$.

   (c) $\Sigma^{\Sigma_n}(\lambda)$ is closed under propositional connectives $\lor,\land$.

   (d) Suppose $\forall x\in d\,A(x)\not\in\Delta_0$. Then $\forall x\in d\,A(x)\in\Sigma^{\Sigma_n}(\lambda)$ iff $A(\emptyset)\in\Sigma^{\Sigma_n}(\lambda)$ and $\mathrm{rk}_L(d)<\lambda$.

   (e) Suppose $\exists x\in d\,A(x)\not\in\Delta_0$. Then $\exists x\in d\,A(x)\in\Sigma^{\Sigma_n}(\lambda)$ iff $A(\emptyset)\in\Sigma^{\Sigma_n}(\lambda)$ and $\mathrm{rk}_L(d)\leq\lambda$.

4. For a Σ_1-formula $A(x)$, $\exists x\in P_{\rho_0}\,A(x)$ is a $\Sigma_1(P_{\rho_0})$*-formula.*

Note that the predicates P,P_{ρ_0} do not occur in Σ_m-formulae.

Definition 5.7 Let us extend the domain $dom(F_x)=\mathrm{Hull}(x)$ of the Mostowski collapse to formulae.

$$dom(F_x)=\{A\in\Sigma_1\cup\Pi_1:\mathsf{k}(A)\subset\mathrm{Hull}(x)\}.$$

For $A\in dom(F_x)$, F_x"A denotes the result of replacing each constant $c\in L_{\rho_0}$ by $F_x(c)$, each unbounded existential quantifier $\exists z\in L_{\rho_0}$ by $\exists z\in L_{F_x(\rho_0)}$, and each unbounded universal quantifier $\forall z\in L_{\rho_0}$ by $\forall z\in L_{F_x(\rho_0)}$.

For sequent, i.e., finite set of sentences $\Gamma\subset dom(F_x)$, put F_x"$\Gamma=\{F_x$"$A: A\in\Gamma\}$.

The assignment of disjunctions $A\simeq\bigvee(A_\iota)_{\iota\in J}$ or conjunctions $A\simeq\bigwedge(A_\iota)_{\iota\in J}$ to sentences A is defined as in [3] *except* for $\Sigma_1\cup\Pi_1$-sentences.

Definition 5.8 *1.* If M is one of the literals $a\in b, a\not\in b$, then for $J:=0$

$$M:\simeq\begin{cases}\bigvee(A_\iota)_{\iota\in J} & \text{if } M \text{ is false (in } L)\\ \bigwedge(A_\iota)_{\iota\in J} & \text{if } M \text{ is true}\end{cases}$$

2. $(A_0\lor A_1):\simeq\bigvee(A_\iota)_{\iota\in J}$ and $(A_0\land A_1):\simeq\bigwedge(A_\iota)_{\iota\in J}$ for $J:=2$.

3. $P(b,c):\simeq\bigvee(0\not\in 0)_{\iota\in J}$ and $\neg P(b,c):\simeq\bigwedge(0\in 0)_{\iota\in J}$ with

$$J:=\begin{cases}1 & \text{if } \exists\alpha[b=\Psi_{\omega_1}(\alpha)\,\&\,c=F_{b\cup\{\omega_1\}}(\rho_0)]\\ 0 & \text{otherwise}\end{cases}.$$

4. $P_{\rho_0}(a):\simeq\bigvee(0\not\in 0)_{\iota\in J}$ and $\neg P_{\rho_0}(a):\simeq\bigwedge(0\in 0)_{\iota\in J}$ with

$$J:=\begin{cases}1 & \text{if } \exists\alpha[a=\Psi_{\rho_0}(\alpha)]\\ 0 & \text{otherwise}\end{cases}.$$

5. Let $\exists z \in b\, \theta[z] \in \Sigma_0$ for $b \in L_{\rho_0} \cup \{L_{\rho_0}\}$. Then for the set

$$d := \mu z \in b\, \theta[z] := \left\{ \begin{array}{ll} \min_{<_L}\{d : d \in b \wedge \theta[d]\} & \text{if } \exists z \in b\, \theta[z] \\ 0 & \text{otherwise} \end{array} \right.$$

with a canonical well ordering $<_L$ on L, and $J = \{d\}$

$$\begin{array}{rcl} \exists z \in b\, \theta[z] & :\simeq & \bigvee(d \in b \wedge \theta[d])_{d \in J} \\ \forall z \in b\, \neg\theta[z] & :\simeq & \bigwedge(d \in b \to \neg\theta[d])_{d \in J} \end{array}$$

where $d \in b$ denotes a true literal, e.g., $d \notin d$ when $b = L_{\rho_0}$.

6. For a $\Sigma_1(P_{\rho_0})$-sentence $\exists x \in P_{\rho_0}\, A(x)$,

$$\begin{array}{rcl} \exists x \in P_{\rho_0}\, A(x) & \simeq & \bigvee(A(a))_{a \in J} \\ \forall x \in P_{\rho_0}\, \neg A(x) & \simeq & \bigwedge(\neg A(a))_{a \in J} \\ \text{with } J & = & \{a : \exists \alpha (a = \Psi_{\rho_0}(\alpha))\} \end{array}$$

7. Otherwise set for $a \in L_{\rho_0} \cup \{L_{\rho_0}\}$ and $J := \{b : b \in a\}$

$$\exists x \in a\, A(x) :\simeq \bigvee(A(b))_{b \in J} \text{ and } \forall x \in a\, A(x) :\simeq \bigwedge(A(b))_{b \in J}.$$

The rank $\mathrm{rk}(A)$ of sentences A is defined by recursion on the number of symbols occurring in A.

Definition 5.9 *1.* $\mathrm{rk}(\neg A) := \mathrm{rk}(A)$.

2. $\mathrm{rk}(a \in b) := 0$.

3. $\mathrm{rk}(P(b, c)) := \mathrm{rk}(P_{\rho_0}(a)) := 1$.

4. $\mathrm{rk}(A_0 \vee A_1) := \max\{\mathrm{rk}(A_0), \mathrm{rk}(A_1)\} + 1$.

5. $\mathrm{rk}(\exists x \in a\, A(x)) := \max\{\omega\alpha, \mathrm{rk}(A(\emptyset)) + 1\}$ for $\alpha = \mathrm{rk}_L(a)$.

6. $\mathrm{rk}(\exists x \in P_{\rho_0}\, A(x)) = \rho_0$.

Proposition 5.10 *Let* $A \simeq \bigvee(A_\iota)_{\iota \in J}$ *or* $A \simeq \bigwedge(A_\iota)_{\iota \in J}$.

1. $\forall \iota \in J(\mathsf{k}(A_\iota) \subset \mathsf{k}(A) \cup \{\iota\})$.

2. $A \in \Sigma^{\Sigma_n}(\lambda) \Rightarrow \forall \iota \in J(A_\iota \in \Sigma^{\Sigma_n}(\lambda))$.

3. *For an ordinal* $\lambda \leq \rho_0$ *with* $\omega\lambda = \lambda$, $\mathrm{rk}(A) < \lambda \Rightarrow A \in \Sigma^{\Sigma_n}(\lambda)$.

4. $\mathrm{rk}(A) < \rho_0 + \omega$.

5. $\mathrm{rk}(A) \in \{\omega\, \mathrm{rk}_L(a) + i : a \in \mathsf{qk}(A) \cup \{\rho_0\}, i \in \omega\} \subset \mathrm{Hull}(\mathsf{k}(A))$.

6. $\forall \iota \in J(\mathrm{rk}(A_\iota) < \mathrm{rk}(A))$.

5.3 Operator controlled derivations

In the remaining parts of this section we work in the intuitionistic fixed point theory $\mathrm{FiX}^i(T_1)$.

Sequents are finite sets of sentences, and inference rules are formulated in one-sided sequent calculus. In what follows by an operator we mean an $\mathcal{H}_\gamma[\Theta]$ for a finite set Θ of sets.

Definition 5.11 Define a relation $(\mathcal{H},\kappa)\vdash^a_b\Gamma$ as follows.

$(\mathcal{H},\kappa)\vdash^a_b\Gamma$ holds if

$$\{a\}\cup\mathsf{k}(\Gamma)\subset\mathcal{H}:=\mathcal{H}(\emptyset) \tag{6}$$

and one of the following cases holds:

($\bigvee$) $A\simeq\bigvee\{A_\iota:\iota\in J\}$, $A\in\Gamma$ and there exist $\iota\in J$ and $a(\iota)<a$ such that

$$\mathrm{rk}_L(\iota)<\kappa\Rightarrow\mathrm{rk}_L(\iota)<a \tag{7}$$

and $(\mathcal{H},\kappa)\vdash^{a(\iota)}_b\Gamma,A_\iota$.

($\bigwedge$) $A\simeq\bigwedge\{A_\iota:\iota\in J\}$, $A\in\Gamma$ and for every $\iota\in J$ there exists an $a(\iota)<a$ such that $(\mathcal{H}[\{\iota\}],\kappa)\vdash^{a(\iota)}_b\Gamma,A_\iota$.

(*cut*) There exist $a_0<a$ and C such that $\mathrm{rk}(C)<b$ and $(\mathcal{H},\kappa)\vdash^{a_0}_b\Gamma,\neg C$ and $(\mathcal{H},\kappa)\vdash^{a_0}_b C,\Gamma$.

(**P**) There exists $\alpha<\omega_1$ such that $(\exists x,y<\omega_1[\alpha<x\land P(x,y)])\in\Gamma$.

($\mathbf{F}_{x\cup\{\omega_1\}}$) $x=\Psi_{\omega_1}(\beta)\in\mathcal{H}$ for a β and there exist $a_0<a$, $\Gamma_0\subset\Sigma_1$ and Λ such that $\mathsf{k}(\Gamma_0)\subset\mathrm{Hull}(x\cup\{\omega_1\})$, $\Gamma=\Lambda\cup(F_{x\cup\{\omega_1\}}\text{"}\Gamma_0)$ and $(\mathcal{H},\kappa)\vdash^{a_0}_b\Lambda,\Gamma_0$, where $F_{x\cup\{\omega_1\}}$ denotes the Mostowski collapse $F_{x\cup\{\omega_1\}}:\mathrm{Hull}(x\cup\{\omega_1\})\leftrightarrow L_{F_{x\cup\{\omega_1\}}(\rho_0)}$.

($\mathbf{P}_{\rho_0}$) There exists $\alpha<\rho_0$ such that $(\exists x<\rho_0[\alpha<x\land P_{\rho_0}(x)])\in\Gamma$.

($\mathbf{F}_x$) $x=\Psi_{\rho_0}(\beta)\in\mathcal{H}$ for a β and there exist $a_0<a$, $\Gamma_0\subset\Sigma_1$ and Λ such that $\mathsf{k}(\Gamma_0)\subset\mathrm{Hull}(x)$, $\Gamma=\Lambda\cup(F_x\text{"}\Gamma_0)$ and $(\mathcal{H},\kappa)\vdash^{a_0}_b\Lambda,\Gamma_0$, where F_x denotes the Mostowski collapse $F_x:\mathrm{Hull}(x)\leftrightarrow L_{F_x(\rho_0)}$.

(*Ref*) $b\geq\rho_0$, and there exist an ordinal $a_0<a$, a set c and a $\Sigma_1(P_{\rho_0})$-formula $A(x)$ such that $(\mathcal{H},\kappa)\vdash^{a_0}_b\Gamma,\forall x\in c\,A(x)$ and $(\mathcal{H},\kappa)\vdash^{a_0}_b\forall y\exists x\in c\,\neg A^{(y)}(x),\Gamma$, where for $A(x)\equiv(\exists z\in P_{\rho_0}\exists w\,B(x))\,(B\in\Delta_0)$, $A^{(y)}(x):\equiv(\exists z\in P_{\rho_0}\cap y\exists w\in y\,B)$.

Lemma 5.12 (Tautology) *If* $\mathsf{k}(\Gamma\cup\{A\})\subset\mathcal{H}$ *then* $(\mathcal{H},\rho_0)\vdash^{2\mathrm{rk}(A)}_0\Gamma,\neg A,A$.

Lemma 5.13 *Let* $\mathrm{rk}(\forall x\in b\,\varphi[x,c])\leq\rho_0+m$ *for an* $m\geq1$, *and* $\Theta_c=\{\neg\forall y(\forall x\in y\,\varphi[x,c]\to\varphi[y,c])\}$. *Then for any operator* $\mathcal{H}$, *and any* a,c, $(\mathcal{H}[\{c,a\}],\rho_0)\vdash^{\rho_0+2m+2+2\mathrm{rk}_L(a)}_{\rho_0+m+1}\Theta_c,\forall x\in a\,\varphi[x,c]$.

Let

$$(\mathcal{H},\rho_0)\vdash^{<\alpha}_{c}\Gamma \quad :\Leftrightarrow \quad \exists\beta<\alpha[(\mathcal{H},\rho_0)\vdash^{\beta}_{c}\Gamma]$$
$$(\mathcal{H},\rho_0)\vdash^{<\alpha}_{<c}\Gamma \quad :\Leftrightarrow \quad \exists d<c[(\mathcal{H},\rho_0)\vdash^{<\alpha}_{d}\Gamma]$$

Lemma 5.14 *Let A be an axiom in $\mathrm{T}(\omega_1)$ except Foundation axiom schema and Π_1-Collection. Then $(\mathcal{H},\rho_0)\vdash^{<\rho_0+\omega}_{0} A$ for any operator $\mathcal{H}=\mathcal{H}_\gamma$.*

Lemma 5.15 (Embedding)
If $\mathrm{T}(\omega_1)\vdash\Gamma[\vec{x}]$, there are $m,k<\omega$ such that for any $\vec{a}\subset L_{\rho_0}$, $(\mathcal{H}[\vec{a}],\rho_0)\vdash^{\rho_0\cdot 2+k}_{\rho_0+m}$ $\Gamma[\vec{a}]$ for any operator $\mathcal{H}=\mathcal{H}_\gamma$.

Proof.
By Lemma 5.13 we have $(\mathcal{H},\rho_0)\vdash^{\rho_0\cdot 2}_{\rho_0+m+1}\forall u,z(\forall y(\forall x\in y\,\varphi[x,z]\to\varphi[y,z])\to\varphi[u,z])$ for some m. By Lemmata 5.12 and 5.14 it remains to consider instances

$$\forall u\in a\exists v\forall w\,\theta\to\exists c\forall u\in a\exists v\in c\forall w\,\theta$$

of Π_1-Collection, where $\theta\equiv\theta(u,v,w)$ is a Δ_0-formula in the language $\{\in\}$.
First by Lemma 5.14 with axioms (3) and (4) we have

$$(\mathcal{H},\rho_0)\vdash^{\rho_0+\omega}_{\rho_0+1}\forall w\,\theta(u,v,w)\leftrightarrow\exists x\in P_{\rho_0}\tau(x,u,v)$$

where $\tau(x,u,v)\equiv[u,v\in L_x\wedge\forall w\in L_x\,\theta(u,v,w)]$. Hence

$$(\mathcal{H},\rho_0)\vdash^{<\rho_0+\omega\cdot 2}_{<\rho_0+\omega}\neg\forall u\in a\exists v\forall w\,\theta,\forall u\in a\exists x\in P_{\rho_0}\exists v\,\tau(x,u,v)$$

On the other hand we have by Lemma 5.12

$$(\mathcal{H},\rho_0)\vdash^{<\rho_0+\omega}_{0}\neg\exists c\forall u\in a\exists x\in P_{\rho_0}\cap c\exists v\in c\,\tau,\exists c\forall u\in a\exists x\in P_{\rho_0}\cap c\exists v\in c\,\tau$$

Hence by the inference (Ref) for the $\Sigma_1(P_{\rho_0})$-formula $\exists x\in P_{\rho_0}\exists v\,\tau(x,u,v)$, we obtain

$$(\mathcal{H},\rho_0)\vdash^{<\rho_0+\omega\cdot 2}_{<\rho_0+\omega}\neg\forall u\in a\exists v\forall w\,\theta,\exists c\forall u\in a\exists x\in P_{\rho_0}\cap c\exists v\in c\,\tau$$

Therefore $(\mathcal{H},\rho_0)\vdash^{\rho_0+\omega\cdot 2}_{<\rho_0+\omega}\forall u\in a\exists v\forall w\,\theta\to\exists c\forall u\in a\exists v\in c\forall w\,\theta$. □

In the following Lemma 5.16, note that $\mathrm{rk}(\exists x<\omega_1\exists y<\omega_1[\alpha<x\wedge P(x,y)])=\omega_1+1$, and $\mathrm{rk}(\exists x<\rho_0[\alpha<x\wedge P_{\rho_0}(x)])=\rho_0$.

Lemma 5.16 (Predicative Cut-elimination)

1. *If $(\mathcal{H},\kappa)\vdash^{b}_{c+\omega^a}\Gamma\;\&\;[c,c+\omega^a[\cap\{\omega_1+1,\rho_0\}=\emptyset\;\&\;a\in\mathcal{H}\Rightarrow(\mathcal{H},\kappa)\vdash^{\varphi ab}_{c}\Gamma$.*
2. *If $(\mathcal{H}_\gamma,\kappa)\vdash^{b}_{\omega_1+2}\Gamma\;\&\;\gamma\in\mathcal{H}_\gamma\Rightarrow(\mathcal{H}_{\gamma+b},\kappa)\vdash^{\omega^b}_{\omega_1+1}\Gamma$.*
3. *If $(\mathcal{H}_\gamma,\kappa)\vdash^{b}_{\rho_0+1}\Gamma\;\&\;\gamma\in\mathcal{H}_\gamma\Rightarrow(\mathcal{H}_{\gamma+b},\kappa)\vdash^{\omega^b}_{\rho_0}\Gamma$.*

For a formula $\exists x \in d\, A(x)$ and ordinals $\lambda = \mathrm{rk}_L(d), \alpha$, $(\exists x \in d\, A)^{(\exists\lambda|\alpha)}$ denotes the result of restricting the outermost existential quantifier $\exists x \in d$ to $\exists x \in L_\alpha$, $(\exists x \in d\, A)^{(\exists\lambda|\alpha)} \equiv (\exists x \in L_\alpha\, A)$.

Lemma 5.17 (Boundedness) *Let $\lambda \in \{\omega_1, \rho_0\}$, $C \equiv (\exists x \in d\, A) \in \Sigma^{\Sigma_2}(\lambda)$ and $C \notin \{\exists x < \omega_1 \exists y < \omega_1[\alpha < x \land P(x,y)] : \alpha < \omega_1\} \cup \{\exists x < \rho_0[\alpha < x \land P_{\rho_0}(x)] : \alpha < \rho_0\}$.*

1. $(\mathcal{H}, \lambda) \vdash^a_c \Lambda, C \,\&\, a \leq b \in \mathcal{H} \cap \lambda \Rightarrow (\mathcal{H}, \lambda) \vdash^a_c \Lambda, C^{(\exists\lambda|b)}$.
2. $(\mathcal{H}, \kappa) \vdash^a_c \Lambda, \neg C \,\&\, b \in \mathcal{H} \cap \lambda \Rightarrow (\mathcal{H}, \kappa) \vdash^a_c \Lambda, \neg(C^{(\exists\lambda|b)})$.

Lemma 5.18 (Boundedness for $\Sigma_1(P_{\rho_0})$)
Let C be a $\Sigma_1(P_{\rho_0})$-sentence. Then $(\mathcal{H}, \rho_0) \vdash^a_c \Lambda, C \,\&\, a \leq b \in \mathcal{H} \cap \rho_0 \Rightarrow (\mathcal{H}, \rho_0) \vdash^a_c \Lambda, C^{(L_b)}$.

Proof. $C^{L_b} \equiv (\exists z \in P_{\rho_0} \cap L_b \exists w \in L_b\, B)$ when $C \equiv (\exists z \in P_{\rho_0} \exists w\, B)$ with a Δ_0-formula B. The lemma is seen from (7). □

5.4 Collapsing derivations

In this subsection derivations of $\Sigma^{\Sigma_2}(\omega_1)$ sentences are shown to be collapsed to derivations with heights and cut ranks $< \omega_1$.

Lemma 5.19 (Collapsing below ω_1)

Suppose $\gamma \in \mathcal{H}_\gamma[\Theta]$ with $\Theta \subset \mathcal{H}_\gamma(\Psi_{\omega_1}(\gamma))$, and $\Gamma \subset \Sigma^{\Sigma_2}(\omega_1)$.
Then for $b = \Psi_{\omega_1}(\gamma + \omega^{\omega_1 + a})$,

$$(\mathcal{H}_\gamma[\Theta], \omega_1) \vdash^a_{\omega_1+1} \Gamma \Rightarrow (\mathcal{H}_{\gamma+\omega^{\omega_1+a}+1}[\Theta], \omega_1) \vdash^b_b \Gamma.$$

Lemma 5.20 (Collapsing below ρ_0)

Suppose $\gamma \in \mathcal{H}_\gamma[\Theta]$ with $\Theta \subset \mathcal{H}_\gamma(\Psi_{\rho_0}(\gamma))$, and $\Gamma \subset \Sigma^{\Sigma_2}(\rho_0) \cup \Sigma_1(P_{\rho_0})$.
Then for $\hat{a} = \gamma + \omega^{\rho_0 + a}$

$$(\mathcal{H}_\gamma[\Theta], \rho_0) \vdash^a_{\rho_0} \Gamma \Rightarrow (\mathcal{H}_{\hat{a}+1}[\Theta], \rho_0) \vdash^{\Psi_{\rho_0}(\hat{a})}_{\Psi_{\rho_0}(\hat{a})} \Gamma.$$

Proof by induction on a, cf. Lemma 5.1.

First note that $\Psi_{\rho_0}(\hat{a}) \in \mathcal{H}_{\hat{a}+1}[\Theta]$ since $\hat{a} = \gamma + \omega^{\rho_0 + a} \in \mathcal{H}_\gamma[\Theta] \subset \mathcal{H}_{\hat{a}+1}[\Theta]$ by the assumption, $\{\gamma, a\} \subset \mathcal{H}_\gamma[\Theta]$.

Assume $(\mathcal{H}_\gamma[\Theta][\Theta_0], \rho_0) \vdash^{a_0}_{\rho_0} \Gamma_0$ with $\Theta_0 \subset \mathcal{H}_\gamma(\Psi_{\rho_0}(\gamma))$. Then by $\gamma \leq \hat{a}$, we have $\widehat{a_0} \in \mathcal{H}_\gamma[\Theta][\Theta_0] \subset \mathcal{H}_\gamma(\Psi_{\rho_0}(\gamma)) \subset \mathcal{H}_{\hat{a}}(\Psi_{\rho_0}(\hat{a}))$. This yields that

$$a_0 < a \Rightarrow \Psi_{\rho_0}(\widehat{a_0}) < \Psi_{\rho_0}(\hat{a})$$

Second observe that $\mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta] \subset \mathcal{H}_{\hat{a}+1}[\Theta]$ by $\gamma \leq \hat{a} + 1$.

Third we have

$$\mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma(\Psi_{\rho_0}(\gamma))$$

When Γ is one of axioms (**P**) and ($\mathbf{P}_{\rho_0}$), there is nothing to show.

Consider the case when the last inference is a (Ref).

$$\frac{(\mathcal{H}_\gamma[\Theta],\rho_0)\vdash^{a_0}_{\rho_0}\Gamma,\forall x\in c\,A(x)\quad(\mathcal{H}_\gamma[\Theta],\rho_0)\vdash^{a_0}_{\rho_0}\forall y\exists x\in c\,\neg A^{(y)}(x),\Gamma}{(\mathcal{H}_\gamma[\Theta],\rho_0)\vdash^{a}_{\rho_0}\Gamma}\ (Ref)$$

where $a_0<a$ and $A(x)\equiv(\exists z\in P_{\rho_0}\exists w\,B(x))$ is a $\Sigma_1(P_{\rho_0})$-formula with a Δ_0-formula B.

For each $d\in c$ we have by Inversion

$$(\mathcal{H}_\gamma[\Theta\cup\{d\}],\rho_0)\vdash^{a_0}_{\rho_0}\Gamma,A(d)$$

where $c\in\mathcal{H}_\gamma(\Psi_{\rho_0}(\gamma))$. Hence $\mathrm{rk}_L(d)<\mathrm{rk}_L(c)\in\mathcal{H}_\gamma(\Psi_{\rho_0}(\gamma))\cap\rho_0\subset\Psi_{\rho_0}(\gamma)$, and $\mathrm{rk}_L(d)<\Psi_{\rho_0}(\gamma)$. Therefore $d\in\mathcal{H}_\gamma(\Psi_{\rho_0}(\gamma))$. By IH we have for $\widehat{a_0}=\gamma+\omega^{\rho_0+a_0}$ and $\beta_0=\Psi_{\rho_0}(\widehat{a_0})\in\mathcal{H}_{\widehat{a_0}+1}[\Theta]$

$$(\mathcal{H}_{\widehat{a_0}+1}[\Theta\cup\{d\}],\rho_0)\vdash^{\beta_0}_{\beta_0}\Gamma,A(d)$$

Boundedness lemma 5.18 yields

$$(\mathcal{H}_{\widehat{a_0}+1}[\Theta\cup\{d\}],\rho_0)\vdash^{\beta_0}_{\beta_0}\Gamma,A^{(L_{\beta_0})}(d)$$

Since $d\in c$ is arbitrary, we obtain by $(\bigwedge)$

$$(\mathcal{H}_{\widehat{a_0}+1}[\Theta],\rho_0)\vdash^{\beta_0+1}_{\beta_0}\Gamma,\forall x\in c\,A^{(L_{\beta_0})}(x)\tag{8}$$

On the other hand we have by Inversion for $L_{\beta_0}\in\mathcal{H}_{\widehat{a_0}+1}[\Theta]$

$$(\mathcal{H}_{\widehat{a_0}+1}[\Theta],\rho_0)\vdash^{a_0}_{\rho_0}\exists x\in c\,\neg A^{(L_{\beta_0})}(x),\Gamma$$

Since $\exists x\in c\,\neg A^{(L_{\beta_0})}(x)\in\Sigma^{\Sigma_2}(\rho_0)$, IH yields for $\widehat{a_1}=\widehat{a_0}+1+\omega^{\rho_0+a_0}=\gamma+\omega^{\rho_0+a_0}\cdot 2$ and $\beta_1=\Psi_{\rho_0}(\widehat{a_1})$

$$(\mathcal{H}_{\widehat{a_1}+1}[\Theta],\rho_0)\vdash^{\beta_1}_{\beta_1}\exists x\in c\,\neg A^{(L_{\beta_0})}(x),\Gamma\tag{9}$$

We have $\mathrm{rk}(\forall x\in c\,A^{(L_{\beta_0})}(x))\in\mathrm{Hull}(\mathsf{k}(\forall x\in c\,A^{(L_{\beta_0})}(x)))\cap\rho_0\subset\mathcal{H}_{\widehat{a_0}+1}[\Theta]\cap\rho_0\subset\mathcal{H}_{\widehat{a_0}+1}(\Psi_{\rho_0}(\gamma))\cap\rho_0\subset\Psi_{\rho_0}(\hat{a})$ by Proposition 5.10.5.

By a (cut) with (8) and (9) we obtain with $\Psi_{\rho_0}(\hat{a})>\beta_1>\beta_0$

$$(\mathcal{H}_{\hat{a}+1}[\Theta],\rho_0)\vdash^{\Psi_{\rho_0}(\hat{a})}_{\Psi_{\rho_0}(\hat{a})}\Gamma$$

Other case ae seen as in [1]. □

6 Proof of Theorem 1.1

For a sentence $\exists x\in L_{\omega_1}\varphi$ with a Σ_2-formula φ in the language $\{\in,\omega_1\}$, assume $T_1\vdash\exists x\in L_{\omega_1}\varphi$. Then by Lemmata 3.2 and 5.15, pick an $m>0$ such that the

fact $(\mathcal{H}_0, \rho_0) \vdash^{\rho_0 \cdot 2+m}_{\rho_0+m} \exists x \in L_{\omega_1}\varphi$ is provable in $\mathrm{FiX}^i(T_1)$. In what follows work in $\mathrm{FiX}^i(T_1)$. Predicative Cut Elimination 5.16.1 and 5.16.3 yields

$$(\mathcal{H}_\gamma, \rho_0) \vdash^{a}_{\rho_0} \exists x \in L_{\omega_1}\varphi$$

for $\gamma = \omega_{m-1}(\rho_0 \cdot 2 + m)$ and $a = \omega_m(\rho_0 \cdot 2 + m)$. Then Collapsing below ρ_0 5.20 yields

$$(\mathcal{H}_{\omega_{m+1}(\rho_0 \cdot 2+m)+1}, \rho_0) \vdash^{\beta}_{\beta} \exists x \in L_{\omega_1}\varphi$$

for $\gamma + \omega^{\rho_0+a} = \omega_{m+1}(\rho_0 \cdot 2 + m)$ and $\beta = \Psi_{\rho_0}(\omega_{m+1}(\rho_0 \cdot 2 + m))$. Predicative Cut Elimination 5.16.1 and 5.16.2 yields

$$(\mathcal{H}_{\omega_{m+1}(\rho_2 \cdot 2+m)+\varphi\beta\beta}, \rho_0) \vdash^{\varphi\beta\beta}_{\omega_1+1} \exists x \in L_{\omega_1}\varphi$$

for $\omega_1 + 2 + \omega^{\beta} = \beta$ and $\omega^{\varphi\beta\beta} = \varphi\beta\beta$. A fortiori,

$$(\mathcal{H}_{\omega_{m+1}(\rho_2 \cdot 2+m)+\varphi\beta\beta}, \omega_1) \vdash^{\varphi\beta\beta}_{\omega_1+1} \exists x \in L_{\omega_1}\varphi$$

Then Collapsing below ω_1 5.19 yields

$$(\mathcal{H}_{\omega_{m+1}(\rho_2 \cdot 2+m)+(\varphi\beta\beta\cdot)2+1}, \omega_1) \vdash^{\delta}_{\delta} \exists x \in L_{\omega_1}\varphi$$

for $\omega_{m+1}(\rho_2 \cdot 2 + m) + \varphi\beta\beta + \omega^{\omega_1+\varphi\beta\beta} + 1 = \omega_{m+1}(\rho_2 \cdot 2 + m) + (\varphi\beta\beta\cdot)2 + 1$ and $\delta = \Psi_{\omega_1}(\omega_{m+1}(\rho_2 \cdot 2 + m) + (\varphi\beta\beta\cdot)2)$.

Boundedness 5.17.1 yields for $\delta < \Psi_{\omega_1}(\omega_n(\rho_0 + 1))$ with $n = m + 2$

$$(\mathcal{H}_{\omega_n(\rho_0+1)+1}, \omega_1) \vdash^{\delta}_{\delta} \exists x \in L_{\Psi_{\omega_1}(\omega_n(\rho_0+1))}\varphi$$

We see then by transfinite induction up to the countable ordinal δ that inference rules in the controlled derivation of $\exists x \in L_{\Psi_{\omega_1}(\omega_n(\rho_0+1))}\varphi$ with cut rank$< \omega_1$ are $(\bigvee)$, $(\bigwedge)$, (cut), and $(\mathbf{F}_{x\cup\{\omega_1\}})$, and since these inference rules are truth-preserving, we conclude again by transfinite induction up to δ that $\exists x \in L_{\Psi_{\omega_1}(\omega_n(\rho_0+1))}\varphi$ is true.

Since the whole proof is formalizable in $\mathrm{FiX}^i(T_1)$, we conclude $\mathrm{FiX}^i(T_1) \vdash \exists x \in L_{\Psi_{\omega_1}(\omega_n(\rho_0+1))}\varphi$. Finally Theorem 5.2 yields $T_1 \vdash \exists x \in L_{\Psi_{\omega_1}(\omega_n(\rho_0+1))}\varphi$. This completes a proof of Theorem 1.1.

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